WOLFRAM TECHNOLOGY CONFERENC

Using *Mathematica* for the Analysis of Macroeconomic Models

Jan Kodera and Miloslav Vosvrda
Centre for Dynamical Economics and Econometrics in Prague

Abstract (*Section*)

Contemporary economics requires using of modern methods of analysis. Linear and nonlinear dynamic models are applied in economics nearly one hundred years ago. Using non-linear dynamic models is considered as a new qualitative approach, which requires demanding apparatus of non-linear differential equations. As the parameters of the economic models may change we have many variants of solution mentioned non-linear models. So, not only solution of non-linear dynamic models, but also sensitivity of the solution on parameters is the focus of our interest. Graphical image of the results is very important and also very effective for the recognition. Mathematica seems to be convenient tool which fulfills the requirements of researches with best results. In this presentation we would like to exhibit the macroeconomic analysis with help of Mathematica using the example of Goodwin model. A purpose of this paper is to derive Goodwin's model with a specific function for the technological progress. This model contains two differential equations, one for share of labor dynamic and one for rate of employment dynamics. Technological progress is a constant in traditional analysis of Goodwin model but in our presentation is considered as a variable dependent on time which is worth noticing. We present four versions of technological progress and attempt to analyze them. The first version is a traditional concept of stationary technological progress represented by constant rate o technological growth, the second approach is endogenously determined technological progress given by the differential equation for the dynamic of growth

rate of technological progress. This way we get a system of three differential equations which exhibits interesting dynamics. Third version is deterministically perturbed technological growth represented by superposition of positive constant and sinusoidal function. Fourth version is the case of random technological progress given by a stochastic differential equation. Our task is to show behavior of Goodwin model with four versions of technological progress and make graphical illustration of the movement of its variables.

Derivation of Goodwin Model

Types of technological progress

Equilibrium (steady state)

- Equations (4), (5), and (6) have in steady state the following form:

$$0 = \gamma[\overline{v} - \rho] - \overline{g},$$
$$0 = \frac{1-\overline{u}}{\sigma} - \overline{g} - \beta,$$
$$0 = \overline{g} - \alpha.$$

  Barred symbols denote steady state values. A solution of the above system of equations has the following form:

$$\overline{v} = \frac{\gamma\rho+\alpha}{\gamma}, \quad \overline{u} = 1 - \sigma(\alpha+\beta), \qquad \overline{g} = \alpha.$$

-

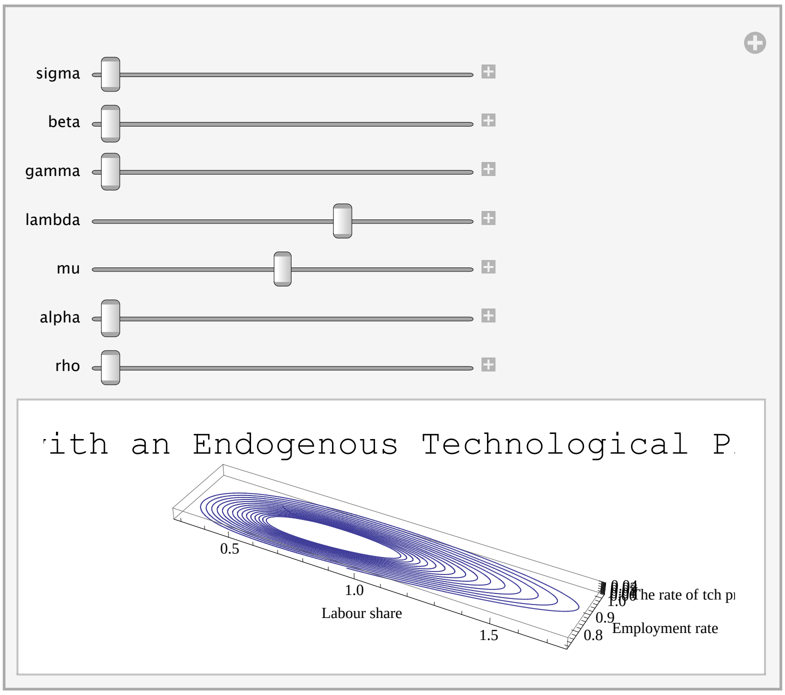
Examples (*Section*)

Let us introduce the numerical example with the following changes in the intervals of parameters:
Let $\alpha\in[0.04, 0.06]$, $\beta\in[0.0, 0.02]$, $\rho\in[0.85, 1]$, $\sigma\in[11,13]$, $\kappa \in [0, 0.1]$, $\lambda\in[0,08]$, $\mu\in[0,0.5]$, $\gamma\in[2, 9]$. The values of these parameters are chosen in accordance with economic nature excluding parameters κ, λ, μ that are optional on important

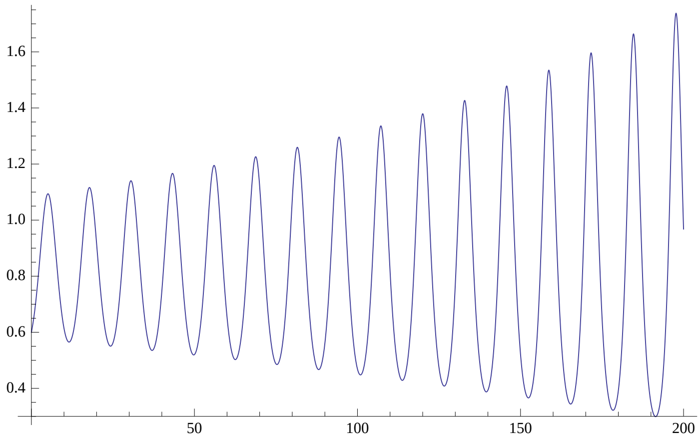
task how to disturb the rate of growth of technological progress described by equation (6) and parameter $\gamma \in [2, 9]$ which gives the speed of adaptation of the rate of employment to a natural rate of employment. The initial values are also given with respect to an economic empirical experience as follows:
u(0)= 0.5, v(0)=0.90, g(0)=0.05.
In the following example the variables u, v, g are firstly denoted by x1, x2, x3 and afterwards for the creation of interpolating functions are labeled again by u, v, g.

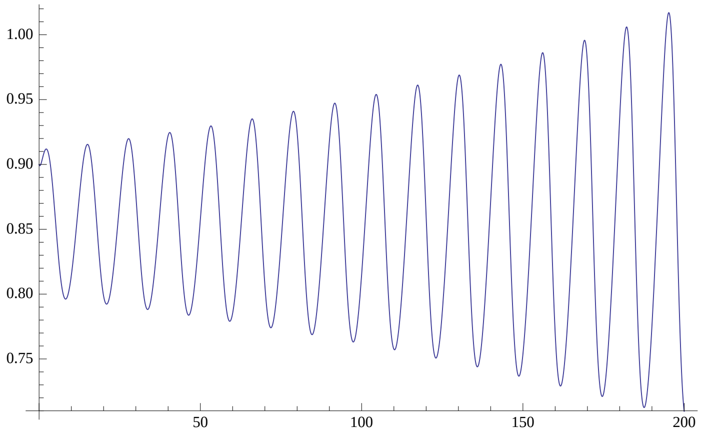
- This item is an input cell computing the example for the endogenously deterministical growth rate of technological progress

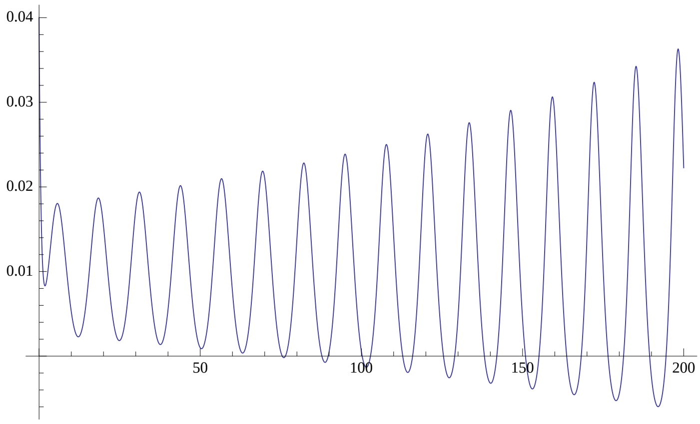
```
beta = 0.01; Clear[x1, x2, x3, v1, u1, a]
Manipulate[
 solution1 =
  NDSolve[{x1'[t] == (gamma * (x2[t] - rho) - x3[t]) * x1[t],
    x2'[t] == (((1 - x1[t]) / sigma) - x3[t] - beta) * x2[t],
    x3'[t] == lambda * (x3[t]) + mu (x1[t] - 1 + sigma (alpha + beta))
    x1[0] == 0.6, x2[0] == 0.9, x3[0] == alpha},
   {x1, x2, x3}, {t, 0, 200}, MaxStepSize -> 0.5];
 ParametricPlot3D[Evaluate[{x1[t], x2[t], x3[t]} /. solution1[[1]]],
  {t, 0, 200}, PlotRange -> All,
  PlotLabel -> "Model with an Endogenous Technological Progress",
  AxesLabel -> {"Labour share", "Employment rate", "The rate of tch progress"}],
 {sigma, 10, 11}, {beta, 0.01, 0.03}, {gamma, 2.75, 2.95},
 {lambda, -4.9, 0}, {mu, 0., 0.1},
 {alpha, 0.04, 0.06}, {rho, 0.85, 0.95}]
```




```
Plot[Evaluate[x1[t] /. solution1[[1]]], {t, 0, 200}]
Plot[Evaluate[x2[t] /. solution1[[1]]], {t, 0, 200}]
Plot[Evaluate[x3[t] /. solution1[[1]]], {t, 0, 200}, PlotRange -> All]
Clear[g];
g[t_] = x3[t] /. solution1[[1]]
Clear[a];
s1 = NDSolve[{a'[t] == g[t], a[0] == 0}, a, {t, 0, 200}]
Plot[Evaluate[a[t]] /. s1[[1]], {t, 0, 200}, PlotRange -> All]
```

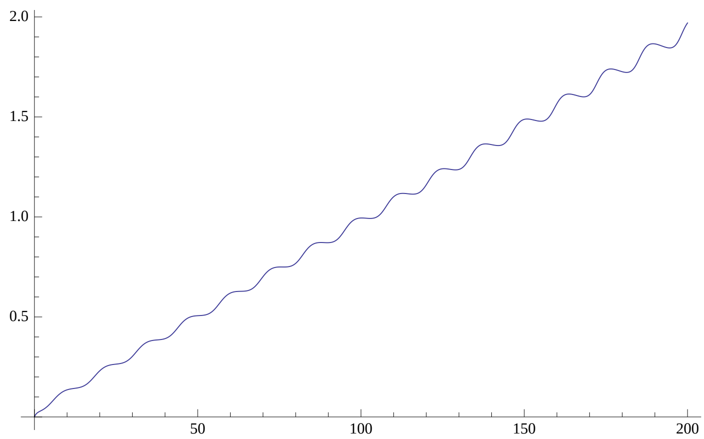






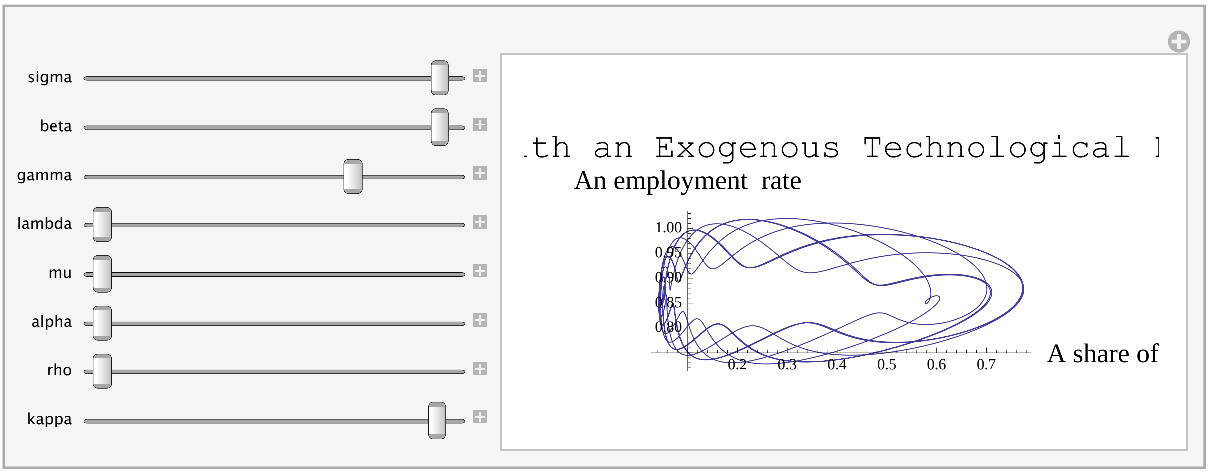

```
InterpolatingFunction[{{0., 200.}}, <>][t]
```

```
{{a → InterpolatingFunction[{{0., 200.}}, <>]}}
```

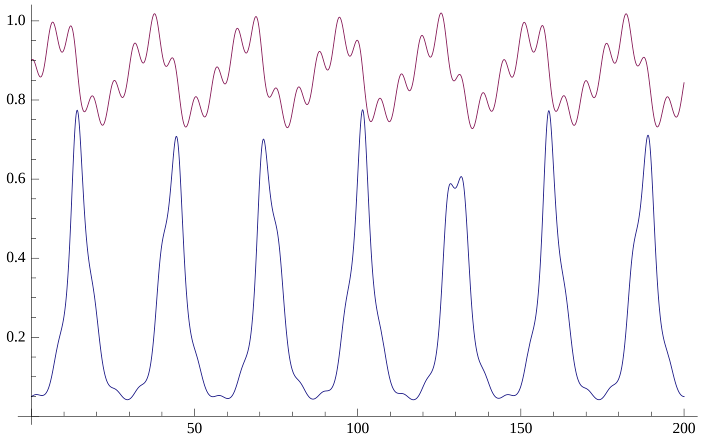


- This item is an input cell computing the example for the exogenously deterministical growth rate of technological progress

```
Clear[x1, x2, x3];
Manipulate[
 solution2 = NDSolve[
  {x1'[t] == (gamma * (x2[t] - rho) - alpha - kappa * Sin[t]) * x1[t]
   x2'[t] == (((1 - x1[t]) / sigma) - alpha - kappa * Sin[t] - beta)
    x2[t], x1[0] == 0.05, x2[0] == 0.90}, {x1, x2}, {t, 0, 200}];
 ParametricPlot[Evaluate[{x1[t], x2[t]} /. solution2[[1]]],
  {t, 0, 200}, PlotRange → All,
  AxesLabel → {"A share of labor", "An employment  rate"},
  PlotLabel → "Model with an Exogenous Technological Progress"],
 {sigma, 10, 11}, {beta, 0.01, 0.03}, {gamma, 2.75, 2.95},
 {lambda, -0.6, 0}, {mu, 0., 0.1}, {alpha, 0.04, 0.06},
 {rho, 0.85, 0.95}, {kappa, 0, 0.05}]
```

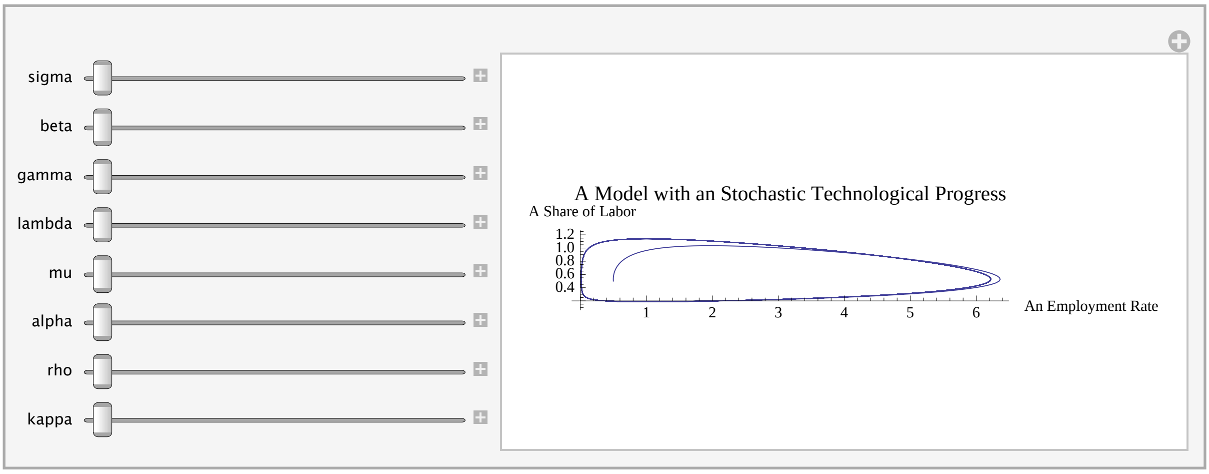


```
Plot[Evaluate[{x1[t], x2[t]} /. solution2[[1]]], {t, 0, 200}]
```

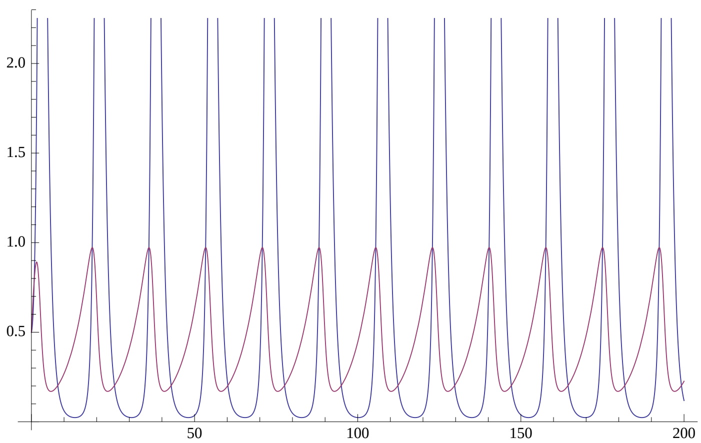


- This item is an input cell computing the example for the endogenously stochastical growth rate of technological progress

```
Clear[x1, x2, x3]
Manipulate[
 solution3 = NDSolve[{x1'[t] == (gamma * (x2[t] - rho) +
       Exp[(lambda + mu (1 - x1[t]) + (1 / 2) × sigma^2 × (alpha + beta) ^2) × t +
          sigma × (alpha + beta) × (RandomReal[{-1, 1}] - RandomReal[{-1, 1}])]) * x1[t], x2'[t] ==
     (((1 - x1[t]) / sigma) + Exp[(lambda + mu (1 - x1[t]) + (1 / 2) ×
           sigma^2 × (alpha + beta) ^2) × t + sigma × (alpha + beta) ×
          (RandomReal[{-1, 1}] - RandomReal[{-1, 1}])]) * x2[t],
    x1[0] == 0.5, x2[0] == 0.5}, {x1, x2}, {t, 0, 200},
   MaxStepSize → 0.5];
 ParametricPlot[Evaluate[
   {x1[t], x2[t]} /. solution3[[1]]],
  {t, 0, 200}, PlotRange →
   All, AxesLabel -> {"
An Employment Rate", " A Share of Labor", "A Stochastic Technological Influence"},
  PlotLabel → "A Model with an Stochastic Technological Progress"], {sigma, 6, 8}, {beta, 0.01, 0.03},
 {gamma, 2.75, 2.95}, {lambda, -1.1, 0}, {mu, 0., 0.1},
 {alpha, 0.04, 0.06}, {rho, 0.85, 0.95}, {kappa, 0, 0.1}]
```




```
Plot[Evaluate[{x1[t], x2[t]} /. solution3[[1]]], {t, 0, 200}]
```



Analysis of production and capital (*Section*)

In the above text the movement of $u(t)$ and $v(t)$ were displayed. Now, let us see on the evolution of production and capital.
Using (1) we obtain

$y(t) = e^{a(t)}\, l(t) = e^{a(t)+\beta \mathrm{t}}\, v(t)$,
where we use the following formulae

$$v(t) = \frac{l(t)}{n(t)},\ \ n(t) = e^{\beta \mathrm{t}},\ \ a(t) = g(t)\,.$$

From (1) we have $k\,(t) = \sigma \mathrm{y}\,(t)$.

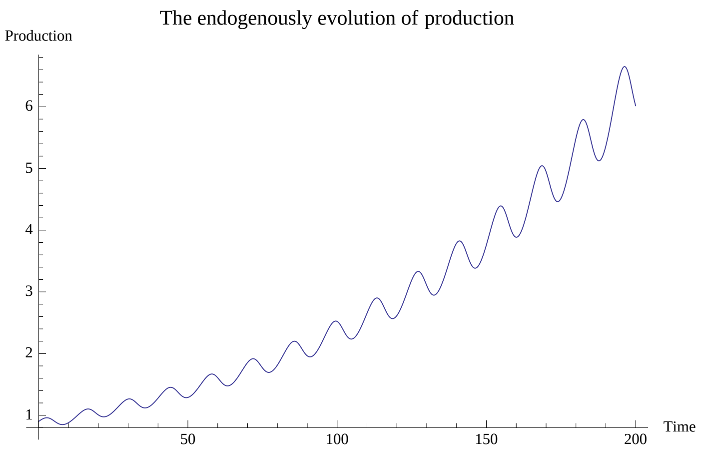
- This item is an input cell computing the example for the endogenously deterministical growth rate of technological progress on an generation of the production and capital

```
beta = 0.01
sigma = 10
```

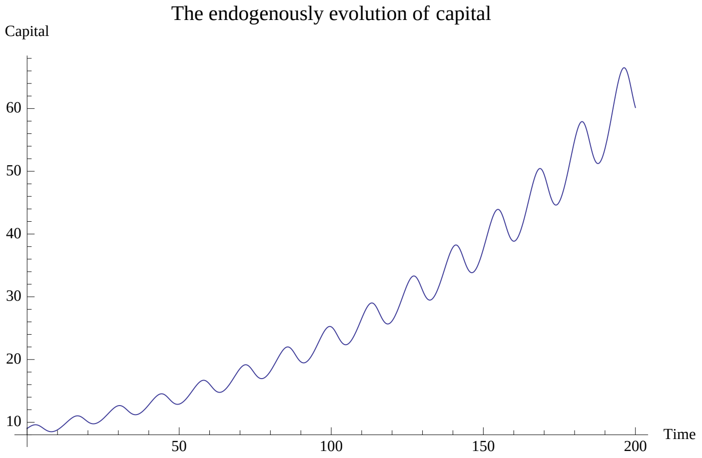
```
0.01
```

```
10
```

```
Plot[Exp[Evaluate[a[t] /. s1[[1]]] + beta * t] * Evaluate[x2[t] /. solution1[[1]]], {t, 0, 200},
 PlotRange → All, AxesLabel → {"Time", "Production"}, PlotLabel → "The endogenously evolution of production"]
```



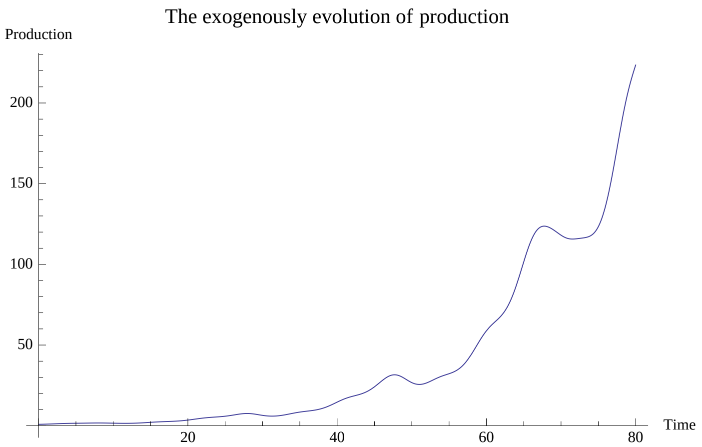
```
Plot[sigma * Exp[Evaluate[a[t] /. s1[[1]]] + beta * t] * Evaluate[x2[t] /. solution1[[1]]], {t, 0, 200},
 PlotRange → All, AxesLabel → {"Time", "Capital"}, PlotLabel → "The endogenously evolution of capital"]
```



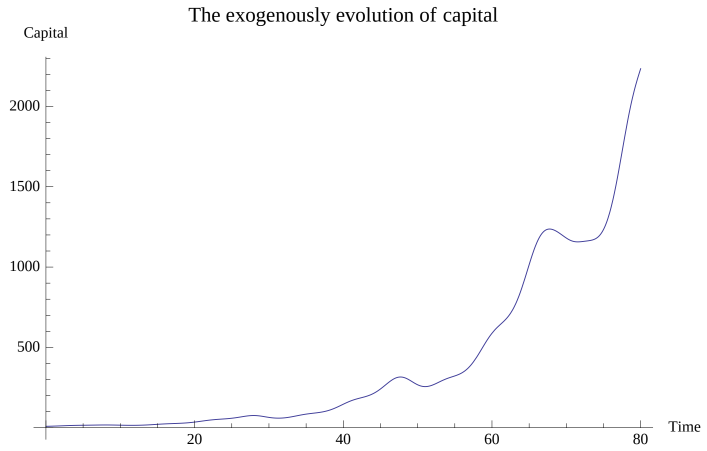
- This is an input cell computing the example for the exogenously determined growth rate of technological progress on an generation of the production and capital

```
alpha = 0.04;
kappa = 0.05;
beta = 0.03;
sigma = 10;
```

```
Plot[Exp[(alpha + beta) * t - kappa * Cos[t]] * Evaluate[x2[t] /. solution2[[1]]], {t, 0, 80},
 PlotRange → All, AxesLabel → {"Time", "Production"}, PlotLabel → "The exogenously evolution of production"]
```



```
Plot[sigma * Exp[(alpha + beta) * t - kappa * Cos[t]] * Evaluate[x2[t] /. solution2[[1]]], {t, 0, 80},
 PlotRange → All, AxesLabel → {"Time", "Capital"}, PlotLabel → "The exogenously evolution of capital"]
```



- This is an input cell computing the example for the endogenously stochastical growth rate of technological progress on an generation of the production and capital

```
calcSolution[gamma_, rho_, lambda_, sigma_, alpha_, beta_, mu_]
 NDSolve[{x1'[t] == (gamma * (x2[t] - rho) +
     Exp[(lambda + mu (1 - x1[t]) + (1 / 2) × sigma^2 × (alpha + beta)^2) × t +
      sigma × (alpha + beta) × (RandomReal[{-10, 10}] - RandomReal[{-10, 10}])]) * x1[t], x2'[t] ==
   (((1 - x1[t]) / sigma) + Exp[(lambda + mu (1 - x1[t]) + (1 / 2) ×
        sigma^2 × (alpha + beta)^2) × t + sigma × (alpha + beta) ×
       (RandomReal[{-1, 1}] - RandomReal[{-1, 1}])]) * x2[t],
  x1[0] == 0.5, x2[0] == 0.5}, {x1, x2}, {t, 0, 200},
 MaxStepSize → 0.5]
```

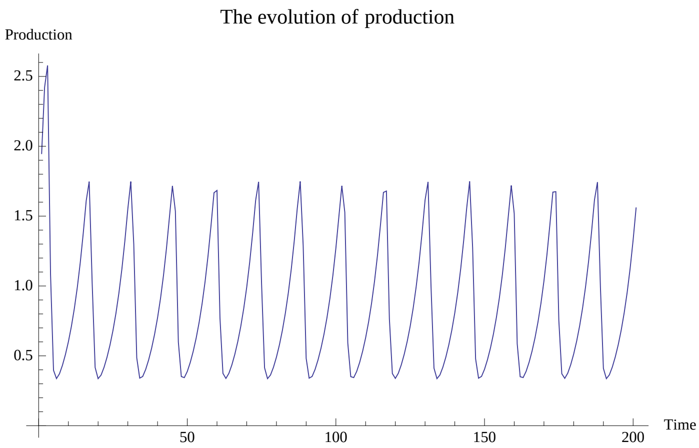
```
gamma = 2.75;
sigma = 6;
lambda = -0.6;
mu = 0;
alpha = 0.04;
beta = 0.01;
rho = 0.85;
kappa = 0.01;
```

```
solution = calcSolution[gamma, rho, lambda, sigma, alpha, beta, mu];
```

```
ggg = Table[Exp[(lambda + mu (1 - Evaluate[x2[t] /. solution[[1]]]) + 1 / 2 sigma^2 (alpha + beta)^2) t + sigma (alpha + beta)],
   {t, 0, 200}];
```

```
y = Table[Exp[ggg[[t + 1]] + beta] * Evaluate[x2[t] /. solution[[1]]], {t, 0, 200}];
```

```
ListPlot[y, Joined → True, PlotRange → All, AxesLabel → {"Time", "Production"}, PlotLabel → "The evolution of production"]
```




```
ListPlot[ggg, Joined → True, PlotRange → All, AxesLabel → {"Time", "Rate of growth"},
 PlotLabel → "The evolution of a growth rate of the technological progress"]
```